Confidence Intervals for the Magnitude of the Largest Aftershock

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Abstract

Aftershock sequences, which follow large earthquakes, last hundreds of days and are characterized by well defined frequency-magnitude and spatio-temporal distributions. The largest aftershocks in a sequence constitute significant hazard and can inflict additional damage to infrastructure. Therefore, the estimation of the magnitude of possible largest aftershocks in a sequence is of high importance. In this work, I propose a statistical model based on Bayesian analysis and extreme value statistics to describe the distribution of magnitudes of the largest aftershocks in a sequence. I derive an analytical expression for a Bayesian predictive distribution function for the magnitude of the largest expected aftershock and compute the corresponding confidence intervals. I assume that the occurrence of aftershocks can be modeled, to a good approximation, by a non-homogeneous Poisson process with a temporal event rate given by the modified Omori law. I also assume that the frequency-magnitude statistics of aftershocks can be approximated by Gutenberg-Richter scaling. I apply the analysis to 30 prominent aftershock sequences in order to compute the Bayesian predictive distributions and the corresponding confidence intervals. In the analysis, I use the information of the early aftershocks in the sequences (in the first $T = 1$, 10, 30 days after the main shock) to estimate retrospectively the confidence intervals for the magnitude of the expected largest aftershocks (*Shcherbakov*, 2014).

1 Introduction

▶ **The purpose of this work:**

- To derive the expression for the Bayesian predictive distribution for the magnitude of the largest aftershock given the information of early aftershocks.
- To assess the applicability of the method in the studies of aftershock sequences and earthquake hazard associated with largest aftershocks.

2 Bayesian predictive distribution

▶ The posterior distribution function for the model parameters $\{\theta, \omega\}$ given the information for the magnitudes and times of early observed aftershocks, $m_i(t_i)$:

$$p(\theta, \omega | \{m_i(t_i)\}) \propto L(\{m_i(t_i)\} | \theta, \omega)\, \pi(\theta, \omega),$$

where $\pi(\theta, \omega)$ is a prior distribution for the model parameters describing the frequency-magnitude statistics and temporal decay rates.

▶ The likelihood function $L(\{m_i(t_i)\} | \theta, \omega)$, assuming that the occurrence of aftershocks follows a non-homogeneous Poisson process (*Utsu et al.*, 1995; *Shcherbakov et al.*, 2005) with magnitudes distributed according to $F_\theta(m)$, has the form (*Daley and Vere-Jones*, 2003):

$$L(\{m_i(t_i)\} | \theta, \omega) = e^{-\Lambda_\omega(T)} \prod_{i=1}^{n} \lambda_\omega(t_i) \prod_{i=1}^{n} f_\theta(m_i),$$

where $f_\theta(m)$ is the probability density function corresponding to $F_\theta(m)$.

▶ Using the order statistics of extremes the probability that the maximum observed event $\mu_k = \max_i \{m_i\}$ is smaller or equal to m is (*Daley and Vere-Jones*, 2003):

$$\Pr(\mu_k \le m | \theta) = [F_\theta(m)]^k.$$

▶ The probability of observing k events on the interval $(T, T+\Delta T]$, assuming that they are generated by a NHP process with the productivity $\Lambda_\omega(\Delta T) = \int_T^{T+\Delta T} \lambda_\omega(t)\, dt$, is (*Daley and Vere-Jones*, 2003):

$$\Pr(k | \Delta T, \omega) = \frac{[\Lambda_\omega(\Delta T)]^k}{k!} e^{-\Lambda_\omega(\Delta T)}.$$

▶ Applying the total probability theorem one can compute the probability distribution that all magnitudes m_i are less than or equal to m for all possible number of events n (*Daley and Vere-Jones*, 2003; *Zöller et al.*, 2013):

$$P(m_{\max} \le m | \theta, \omega) = \sum_{k=0}^{\infty} \frac{[\Lambda_\omega(\Delta T)]^k}{k!} e^{-\Lambda_\omega(\Delta T)} [F_\theta(m)]^k = \exp\{-\Lambda_\omega(\Delta T)\,[1 - F_\theta(m)]\}.$$

▶ The Bayesian predictive distribution for the largest aftershock $m_{\max}$ given the information for the magnitudes and times of each n observed aftershocks $m_i(t_i)$ (*Zöller et al.*, 2013; *Shcherbakov*, 2014):

$$P_B(m_{\max} \le m | \{m_i(t_i)\}) = \int_\Omega \int_\Theta P(m_{\max} \le m | \theta, \omega)\; p(\theta, \omega | \{m_i(t_i)\})\, d\theta\, d\omega,$$

where Θ and Ω define the multidimensional domains of frequency-magnitude distribution and rate parameters.

▶ The Bayesian predictive distribution can be written in the following form (*Shcherbakov*, 2014):

$$P_B(m_{\max} \le m | \{m_i(t_i)\}) \propto \int_\Omega \int_\Theta e^{-\Lambda_\omega(T) - \Lambda_\omega(\Delta T)[1 - F_\theta(m)]} \prod_{i=1}^{n} \lambda_\omega(t_i) f_\theta(m_i)\, d\theta\, d\omega,$$

where I used a noninformative prior distribution, $\pi(\theta, \omega) = \text{const.}$, for the model parameters.

▶ The corresponding probability density function $p_B(m) = \frac{d}{dm} P_B$ is:

$$p_B(m | \{m_i(t_i)\}) \propto \int_\Omega \int_\Theta \Lambda_\omega(T) f_\theta(m)\; e^{-\Lambda_\omega(T) - \Lambda_\omega(\Delta T)[1 - F_\theta(m)]} \prod_{i=1}^{n} \lambda_\omega(t_i) f_\theta(m_i)\, d\theta\, d\omega.$$

With proper normalization the equation gives the distribution for the magnitude of the largest possible aftershock given the magnitudes and times $\{m_i(t_i)\}$ of early n aftershocks.

3 Aftershock model

▶ To be more specific, I assume that aftershock magnitudes are distributed according to the exponential distribution with the probability density $f_\theta(m) = \beta \exp[-\beta (m - m_0)]$ for $m_0 \le m$, where $\theta = \{\beta\}$ and $\beta = \ln(10) b$ where the parameter b is the b-value of the Gutenberg-Richter scaling relation (*Gutenberg and Richter*, 1954). m_0 is the lower magnitude threshold set above the catalog completeness level. I assume that m_0 is given. The corresponding distribution function is $F_\theta(m) = 1 - \exp[-\beta (m - m_0)]$.

▶ The aftershock rate is approximated by the modified Omori law:

$$\lambda_\omega(t) = \frac{1}{\tau} \frac{1}{(1 + t/c)^p}.$$

▶ To fix the future time horizon $T + \Delta T$ I assume that $\Lambda_\omega(\Delta T) = \Lambda_\omega(T)$. From this one can obtain $\Delta T = \left[2(c+T)^{1-p} - c^{1-p}\right]^{1/(1-p)} - (c+T)$.

▶ The Bayesian predictive probability density function for the largest aftershock can be written (*Shcherbakov*, 2014):

$$p_B(m | \{m_i(t_i)\}) \propto \int_0^\infty \int_0^\infty \int_{c_{\min}}^{c_{\max}} \int_{p_{\min}}^{p_{\max}} \Lambda_\omega(T)\, \beta\, e^{-\beta (m - m_0)} \times e^{-\Lambda_\omega(T)\{1 + \exp[-\beta (m - m_0)]\}} \beta^n e^{-\beta n (\bar{m} - m_0)} \times \prod_{i=1}^{n} \lambda_\omega(t_i)\, d\beta\, d\tau\, dc\, dp,$$

where $\bar{m} = \frac{1}{n} \sum_{i=1}^{n} m_i$ is the average magnitude.

▶ The only relevant part of the density can be written in the following form (*Shcherbakov*, 2014):

$$p_B(m | \{m_i(t_i)\}) \propto \int_0^\infty \psi(\beta, m)\, d\beta,$$

where

$$\psi(\beta, m) = \beta^{n+1} e^{-\beta (m - m_0)} \left[\frac{e^{-\beta (\bar{m} - m_0)}}{1 + e^{-\beta (m - m_0)}}\right]^n.$$

▶ To normalize the distribution I assume that the maximum future aftershock is bounded between the maximum observed aftershock μ_n and the upper cutoff magnitude M (*Shcherbakov*, 2014):

$$p_B(m | \{m_i(t_i)\}) = \frac{\int_0^\infty \psi(\beta, m)\, d\beta}{\int_{\mu_n}^{M} \int_0^\infty \psi(\beta, m')\, d\beta\, dm'}.$$

▶ A confidence interval at a given confidence level α for the magnitude of the largest expected aftershock can be computed numerically from the Bayesian predictive distribution (*Shcherbakov*, 2014):

$$\int_{\mu_n}^{m_\alpha} p_B(m | \{m_i(t_i)\})\, dm = 1 - \alpha.$$

This provides a numerical estimate for the upper quantile m_α. Therefore, with probability $100(1 - \alpha)\%$ the largest aftershock is expected to be in the interval $[\mu_n, m_\alpha]$.

4 The $M_W\,7.8$ Nepal earthquake (April 25, 2015)

▶ Bayesian predictive distribution for the magnitude 7.8 Nepal earthquake (April 25, 2015). Each curve corresponds to different early time intervals ($T = 1$, 15, 30 days) after the main shock. The densities are normalized to have a unit area under the curves between the lower μ_n and upper $M = 7.8$ magnitude cutoffs. The 95% confidence intervals are estimated and the upper quantiles are marked at the following magnitudes $m_\alpha = 7.59$, 7.62, and 7.76.

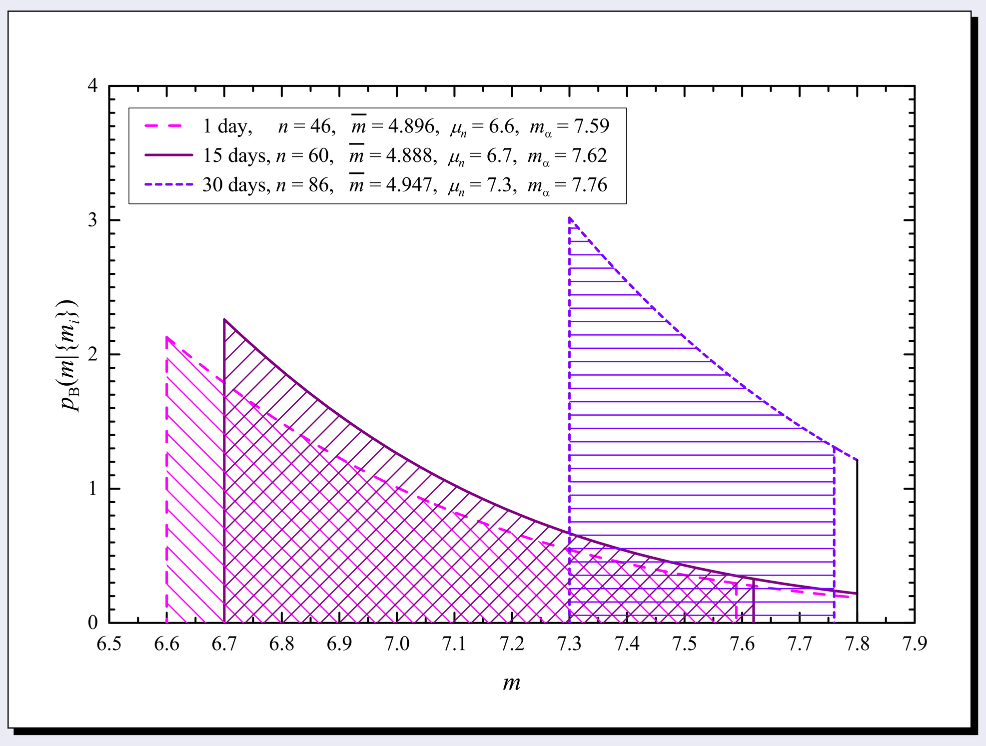

Figure 1. Bayesian predictive distributions for the magnitude 7.8 Nepal earthquake (April 25, 2015) with $m_0 = 4.5$.

▶ The first large aftershock occurred during the first day and had a magnitude 6.6. Another more powerful aftershock of magnitude 6.7 occurred on the second day. And finally the largest aftershock in the sequence (magnitude 7.3) occurred on the 17th day. To normalize each probability density I set the lower magnitude threshold at $\mu_n = 6.6$ for $T = 1$ day, $\mu_n = 6.7$ for $T = 15$ days, and $\mu_n = 7.3$ for $T = 30$ days, respectively. The upper threshold was set at $M = 7.8$. To obtain the corresponding confidence intervals I computed the upper quantiles m_α with $\alpha = 0.05$.

The interval between μ_n and m_α defines 95% of the corresponding area under each curve. From the analysis, the aftershock data after the first day were used to constrain the magnitude of the second large aftershock.

5 Past prominent aftershock sequences

▶ I used the Advanced National Seismic System (`http://www.ncedc.org/anss/`) composite catalog to extract 15 aftershock sequences. I also extracted 14 aftershock sequences from the regional catalogs in 1) southern California (`http://www.data.scec.org/research-tools/alt-2011-dd-hauksson-yang-shearer.html`, *Hauksson et al.* (2012)), 2) northern California (`http://quake.geo.berkeley.edu/ncedc/catalog-search.html`), 3) Alaska (`http://www.aeic.alaska.edu/html_docs/db2catalog.html`), and 4) New Zealand (`http://quakesearch.geonet.org.nz/`).

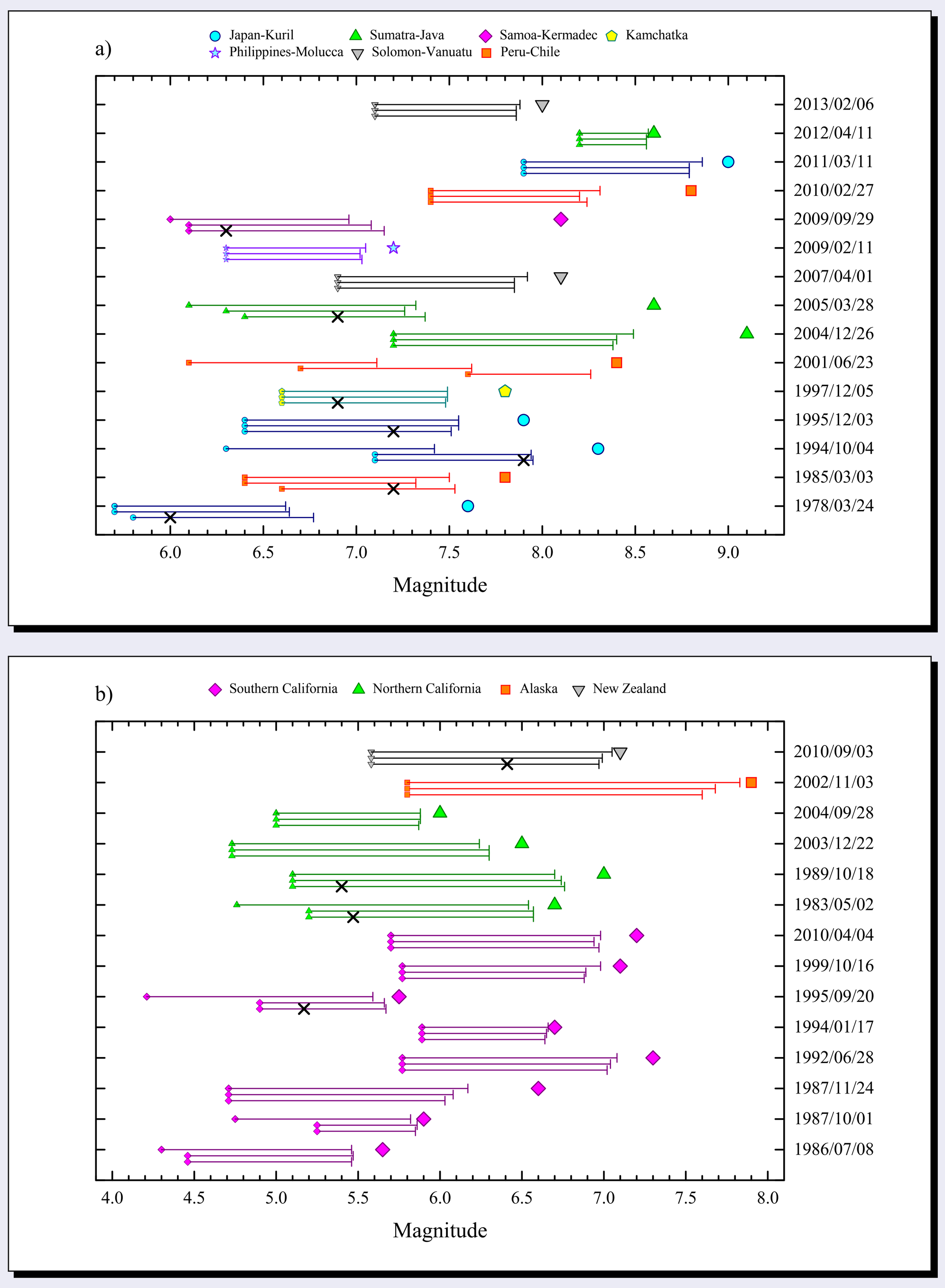

Figure 2. a) Estimated confidence intervals for the largest aftershocks for 15 prominent aftershock sequences. The larger solid symbols indicate the magnitude of the main shocks. The lower bound (smaller solid symbols) is given by the magnitude of the largest detected aftershock μ_n. The upper bound m_α is computed using equation for the confidence interval. For each event the confidence intervals are computed after $T = 1$, 10, and 30 days. The bold cross symbols indicate aftershocks that occurred more than 30 days after the main shocks. b) The confidence intervals for large aftershock sequences in California, New Zealand and Alaska (*Shcherbakov*, 2014).

6 Conclusions

- I proposed a simple statistical model to provide a confidence interval for the magnitude of the largest possible aftershock in a sequence using the early information of already occurred aftershocks. The approach is based on Bayesian analysis and extreme value statistics. It also assumes that aftershock sequences can be modelled as a non-homogeneous Poisson process with a specified frequency-magnitude statistics and the time dependent rate.

References

Daley, D. J., and D. Vere-Jones (2003), *An Introduction to the Theory of Point Processes*, vol. 1, 2nd ed., Springer, New York.

Gutenberg, B., and C. F. Richter (1954), *Seismicity of the Earth and Associated Phenomenon*, 2 ed., Princeton University Press, Princeton.

Hauksson, E., W. Z. Yang, and P. M. Shearer (2012), Waveform relocated earthquake catalog for southern California (1981 to June 2011), *Bull. Seismol. Soc. Am.*, *102*(5), 2239–2244, doi:10.1785/0120120010.

Shcherbakov, R. (2014), Bayesian confidence intervals for the magnitude of the largest aftershock, *Geophys. Res. Lett.*, *41*(18), 6380–6388, doi:10.1002/2014GL061272.

Shcherbakov, R., G. Yakovlev, D. L. Turcotte, and J. B. Rundle (2005), Model for the distribution of aftershock interoccurrence times, *Phys. Rev. Lett.*, *95*, 218,501, doi:10.1103/PhysRevLett.95.218501.

Utsu, T., Y. Ogata, and R. S. Matsu'ura (1995), The centenary of the Omori formula for a decay law of aftershock activity, *J. Phys. Earth*, *43*(1), 1–33.

Zöller, G., M. Holschneider, and S. Hainzl (2013), The maximum earthquake magnitude in a time horizon: Theory and case studies, *Bull. Seismol. Soc. Am.*, *103*(2A), 860–875, doi:10.1785/0120120013.

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